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POTENTIAL SEMI-STABILITY OF p-ADIC ÉTALE COHOMOLOGY

BY

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ABSTRACT

We extend the methods of Faltings and Tsuji, and prove that if K is a field of characteristic 0 with a complete, discrete valuation, and a perfect residue field of characteristic p, then the p-adic étale cohomology of a finite type K-scheme is potentially semi-stable. We prove a similar result for cohomology with compact support, and for cohomology with support in a closed subspace of X. We establish a relationship between these cohomology groups, and the de Rham cohomology of X.

Introduction

Let K be a field of characteristic 0, complete with respect to a discrete valuation, and having perfect residue field of characteristic p. The purpose of this note is to show that the p-adic étale cohomology of an arbitrary finite type K-scheme X is potentially semi-stable. We also prove the analogous fact for p-adic étale cohomology with compact support, and for cohomology with support in an arbitrary closed subset. The case of X proper was previously announced by Tsuji in his ICM talk [Ts]. We also show a relationship between p-adic étale cohomology, and de Rham cohomology along the lines of the de Rham conjecture, including the cases of cohomology with compact support, and with support in a closed set. If X is not smooth, then we need to use Hartshorne's definition [Ha] of de Rham cohomology, and we give an analogous definition of cohomology with compact support.

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Our approach is heavily influenced by that of Tsuji, except that we base our arguments on Faltings' approach to the comparison theorem [Fa]. Faltings' results yield the comparison isomorphism in the case of cohomology, or cohomology with compact support, if X has a sufficiently good compactification to a proper flat scheme over $\mathcal{O}_K$, the ring of integers of K. A refinement of Faltings' arguments shows that the comparison between crystalline and p-adic étale cohomology can be formulated on the level of simplicial schemes. Using work of de Jong [deJ], any scheme X admits a suitable hypercovering by schemes with such good compactifications, and we conclude by a cohomological descent argument. It is in order to apply this method that we need the comparison isomorphism on the level of simplicial schemes. For the case of cohomology with support in a closed set, one uses an excision argument, but as with the descent argument above, this has to be carried out on the level of complexes, and not just on cohomology.

Although a large part of the argument consists of repeating Faltings' techniques in the simplicial, and another slightly more general setting, there are two technical innovations which seem to be worth noting. The first is the definition of the sheaf, denoted by $B_{cr}^+(\bar{\mathcal{O}})$ by Faltings, which is a sort of higher dimensional analogue of Fontaine's ring A_{cris}. Faltings' definition of this sheaf involves a choice of algebraic closure for the function field of the algebraic variety on which one is working. We explain in (2.3) why this is not really necessary. This is important for us, because we want to work simplicially, and it is not clear (at least to the author) that one can make such a choice of algebraic closure sufficiently functorially.

Our second technical contribution is to give a cleaner proof in (2.6) that a certain "almost" map which occurs in the course of the argument is an actual map. The proof of the analogous fact in [Fa3] is more computational.

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§1. Cohomological descent

(1.1) Let D be a small category, and E a D^{op} topos ringed by Λ-algebras, for some ring Λ ([SD, 1.2.1]). We denote by $R\Gamma(E/D, \_)$ the derived functor of the functor from the category of Λ-sheaves on the topos $\underline{\Gamma}(E)$ associated to $E \to D^{\mathrm{op}}$ (denoted $\mathrm{Mod}(\underline{\Gamma}(E), \Lambda)$ in [SD, 1.3.5]) to the category of Λ-sheaves on D^{op} which takes global sections on each fibre of $E \to D^{\mathrm{op}}$.

Examples of the above situation may be obtained as follows: If S is a scheme, and $\pi: \mathcal{E} \to \mathrm{Sch}/S$ is a ringed topos over Sch/S, the category of schemes over S (or possibly some full subcategory), then any functor $F: D^{\mathrm{op}} \to \mathrm{Sch}/S$ gives rise to a ringed D^{op}-topos $\mathcal{E}|_{F(D^{\mathrm{op}})} \to D^{\mathrm{op}}$.

(1.2) Let $m \in \mathbb{N} \cup \{\infty\}$. As usual, we denote by $\Delta[m]$ the category whose objects are ordered sets $[n] = \{0, \ldots, n\}$ with $n \leqslant m$, and whose morphisms are non-decreasing maps. For $i = 0, \ldots, n$, the unique map $[n] \to [n+1]$ which sends i to $i+1$ is denoted by ∂_i, while the natural inclusion is denoted by ∂_{n+1}. We sometimes write Δ for $\Delta[\infty]$.

We denote by $\Delta_1[m]$ the product of $\Delta[m]$ with the category $\{0, 1\}$ consisting of two objects $(0, 1)$ and maps $\psi: 0 \to 1$, id_0, id_1. Thus maps in $\Delta_1[m]$ are of the form (f, id_0) (f, id_1) or (f, ψ) where f is a map in Δ.

(1.3) Let Λ be any ring. The total derived functor of the "global sections" functor Γ taking Λ-sheaves on $\Delta[m]^{\mathrm{op}}$ to Λ-modules, can be explicitly described as follows: If $K^\bullet$ is a complex in $D^+(\Delta[m]^{\mathrm{op}}, \Lambda)$, denote by K^n its value on $[n]$. Then $R\Gamma(K^\bullet)$ is canonically isomorphic to the total complex $T(K^\bullet)$ of

$$K^0 \to K^1 \cdots \to K^n,$$

where $d_i: K^i \to K^{i+1}$ is given by $d_i = \sum_{j=0}^{i} (-1)^j \partial_j$ [SD, 2.3.9].

Similarly, if $K^\bullet$ is in $D^+(\Delta_1[m]^{\mathrm{op}}, \Lambda)$, then $K^\bullet$ consists of two complexes $K_0^\bullet, K_1^\bullet$ on $\Delta[m]$, and a morphism of complexes $K_0^\bullet \to K_1^\bullet$.

We set

$$T_1(K^\bullet) = \mathrm{Cone}(T(K_0^\bullet) \to T(K_1^\bullet)).$$

Note, however, that this is *not* isomorphic to $R\Gamma(K^\bullet)$.

(1.4) Finally, we have an obvious variants of (1.1)–(1.3) for I-adic sheaves. Namely, if Λ is complete with respect to an ideal I, then we can make the above constructions for complexes $K^\bullet$ in $D^+(D^{\mathrm{op}}, \Lambda)$, where this category is defined as the two limit of $D^+(D^{\mathrm{op}}, \Lambda/I^n)$, $n \in \mathbb{N}$. Similarly, if E is ringed by the constant sheaf Λ, then we can define $R\Gamma(E/D, K^\bullet)$ where $K^\bullet$ is a complex in $\varprojlim D^+(E, \Lambda/I^n)$. We refer to [Ek] for more details.

Alternatively, one can work directly with inverse systems of complexes of Λ/I^n-modules $n = 1, 2, \ldots$. Formally this amounts to working in an $\mathbb{N}$-topos $E \times \mathbb{N}$, where the fibre over n is the category of Λ/I^n-sheaves on E. Then $R\Gamma(E/D, K^\bullet)$ can be defined as $R\varprojlim R\Gamma(E \times \mathbb{N}/D \times \mathbb{N}, K^\bullet)$.

(1.5) Denote by L an algebraically closed field of characteristic 0. Consider a scheme X over L, and a proper hypercovering $X^\bullet \to X$ [De, 5.3.8]. For any morphism $Y \to X$ of L-schemes, we set $Y^\bullet = X^\bullet \times_X Y$, the hypercovering of Y obtained by pulling back $X^\bullet$ to Y.

Let Λ be a ring, which is killed by some integer. We will consider three types of data:

Situation 1: Given $X^\bullet$ and X as above, we can regard the category $X^\bullet_{ét}$, whose objects are étale sheaves on $X^i_{ét}$ for some i, as being fibred over Δ^{op} (by sending $X^i_{ét}$ to $[i]$). We write

$$\mathbb{R}\Gamma(X^\bullet, \Lambda) = T(R\Gamma(X^\bullet_{ét}/\Delta, \Lambda)) \tag{1.5.1}$$

and we denote by $H^i(X^\bullet, \Lambda)$ the cohomology of this complex.

Situation 2: Suppose we are in Situation 1, and assume we are given a closed subscheme $Z \subset X$.

Set $U = X \backslash Z$. We regard the category $(X^\bullet, U^\bullet)_{ét}$ whose objects are étale sheaves on X^i and U^i as being fibred over $\Delta_1^{\rm op}$, the fibres over $([i], 0)$ and $([i], 1)$ being $X^i_{ét}$ and $U^i_{ét}$ respectively. Set

$$\mathbb{R}\Gamma_{Z^\bullet}(X^\bullet, \Lambda) = T_1(R\Gamma((X^\bullet, U^\bullet)/\Delta_1, \Lambda)_{et}). \tag{1.5.2}$$

We denote by $H^i_{Z^\bullet}(X^\bullet, \Lambda)$ the cohomology of this complex.

Situation 3: Suppose that there exists a dense open immersion $j\colon X \hookrightarrow \bar{X}$ with $\bar{X}$ proper over L, and a proper hypercovering $\bar{X}^\bullet \to \bar{X}$, such that $\bar{X}^\bullet \times_{\bar{X}} X \xrightarrow{\sim} X^\bullet$ (as hypercoverings of X). Let $Z = \bar{X} - X$, and $Z^\bullet = \bar{X}^\bullet \times_{\bar{X}} Z$. We can regard the category $(\bar{X}^\bullet, Z^\bullet)_{ét}$ whose objects are étale sheaves on the $\bar{X}^i$ and Z^i as being fibred over $\Delta_1^{\rm op}$. (The fibres over $([n], 0)$ and $([n], 1)$ are $\bar{X}^n_{ét}$ and $Z^n_{ét}$ respectively.)

We set

$$\mathbb{R}\Gamma_c(X^\bullet, \Lambda) = T_1(R\Gamma((\bar{X}^\bullet, Z^\bullet)_{ét}/\Delta_1, \Lambda)) \tag{1.5.3}$$

and we denote by $H^i_c(X^\bullet, \Lambda)$ the cohomology of this complex. If $j^\bullet\colon X^\bullet \to \bar{X}^\bullet$ denotes the open immersion of simplicial schemes, we also have an isomorphism

$$\mathbb{R}\Gamma_c(X^\bullet, \Lambda) \xrightarrow{\sim} T(R\Gamma(\bar{X}^\bullet, j^\bullet_! \Lambda)).$$

PROPOSITION (1.6): *With the assumptions of situations 1,2,3 respectively, there are canonical quasi-isomorphisms*

$$R\Gamma(X,\Lambda) \xrightarrow{\sim} \mathbb{R}\Gamma(X^\bullet,\Lambda), \tag{1.6.1}$$

$$R\Gamma_Z(X,\Lambda) \xrightarrow{\sim} \mathbb{R}\Gamma_{Z^\bullet}(X^\bullet,\Lambda), \tag{1.6.2}$$

$$R\Gamma_c(X,\Lambda) \xrightarrow{\sim} \mathbb{R}\Gamma_c(X^\bullet,\Lambda). \tag{1.6.3}$$

Here, the left hand side denotes étale cohomology of X with the indicated support condition.

Proof: The first claim is just usual cohomological descent [SD, 3.3.3, 4.3.2]. For the second claim, note that we have a natural morphism $R\Gamma_Z(X,\Lambda) \to \mathbb{R}\Gamma_{Z^\bullet}(X^\bullet,\Lambda)$, and that this morphism fits into a commutative diagram, whose rows are distinguished triangles

$$\begin{array}{ccccccc} R\Gamma_Z(X,\Lambda) & \longrightarrow & R\Gamma(X,\Lambda) & \longrightarrow & R\Gamma(U,\Lambda) & \longrightarrow & R\Gamma_Z(X,\Lambda)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}\Gamma_{Z^\bullet}(X^\bullet,\Lambda) & \longrightarrow & \mathbb{R}\Gamma(X^\bullet,\Lambda) & \longrightarrow & \mathbb{R}\Gamma(U^\bullet,\Lambda) & \longrightarrow & \mathbb{R}\Gamma_{Z^\bullet}(X^\bullet,\Lambda)[1] \end{array}$$

Since the second and third vertical maps are isomorphisms, so is (1.6.2). Similarly, the third isomorphism follows from the exact triangle

$$R\Gamma_c(X,\Lambda) \to R\Gamma(\bar{X},\Lambda) \to R\Gamma(Z,\Lambda) \to R\Gamma_c(X,\Lambda)[1]. \qquad \blacksquare$$

(1.7) We will want to consider the case where Λ is not a finite group. In particular, suppose p is a prime. We denote by $D^+(X,\mathbb{Q}_p)$ the triangulated category obtained from $D^+(X,\mathbb{Z}_p)$ by formally inverting the maps given by multiplication by p, and we denote by $\otimes^{\mathbb{L}}_{\mathbb{Z}_p}\mathbb{Q}_p\colon D^+(X,\mathbb{Z}_p) \to D^+(X,\mathbb{Q}_p)$ the natural functor. We define

$$R\Gamma(X,\mathbb{Q}_p) = R\varprojlim R\Gamma(X,\mathbb{Z}/p^n) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{Q}_p.$$

We define similarly $R\Gamma_c(X,\mathbb{Q}_p), R\Gamma_Z(X,\mathbb{Q}_p)$ and the obvious variants for hypercoverings, and we denote by $H^i(X,\mathbb{Q}_p)$ etc. the cohomology of the these complexes. We then have

PROPOSITION (1.8): *The statement of Theorem (1.6) holds if we take $\Lambda = \mathbb{Q}_p$.*

Proof: This follows from (1.6) by passing to inverse limits. $\blacksquare$

(1.9) We want to explain descent for de Rham cohomology for proper surjective morphisms. From now on we no longer assume that L is algebraically closed.

For any category of smooth schemes over L, we denote by $\Omega^\bullet$ the sheaf of differentials on the Zariski site of this category. Suppose that in situation 1 of (1.5) the schemes X and X^i are all smooth over L. We set

$$\mathbb{R}\Gamma(X^\bullet, \Omega^\bullet_{X^\bullet}) = T(R\Gamma(X^\bullet_{Zar}/\Delta, \Omega^\bullet)),$$

and denote by $H^i(X^\bullet, \Omega^\bullet_{X^\bullet})$ the cohomology of this complex. Suppose that in situation 2 of (1.5), X and X^i are smooth over L, and that the reduced subschemes Z^{red} and $(Z^i)^{\mathrm{red}}$ are normal crossings divisors. We set

$$R\Gamma_{Z^\bullet}(X^\bullet, \Omega^\bullet_{X^\bullet}) = T_1(R\Gamma((X^\bullet, U^\bullet)_{Zar}/\Delta_1, \Omega^\bullet))$$

and denote by $H^i_{Z^\bullet}(X^\bullet, \Omega^\bullet_{X^\bullet})$ the cohomology of this complex. (We could also have made the above definitions using the infinitesimal sites of $X^\bullet$ and $(X^\bullet, U^\bullet)$, and the structure sheaves on these sites.)

Finally, suppose that in situation 3 of (1.5), the $\bar{X}^i$ are smooth over L, and that the Z^{red} and $(Z^i)^{\mathrm{red}}$ are normal crossings divisors. For each i the cohomology with compact support $\mathbb{R}\Gamma_c(X^i, \Omega^\bullet_{X^i})$ has the following description. Let $\mathcal{I}_{Z^i}$ be the ideal sheaf of Z^i. $\mathcal{I}_{Z^i}$ is equipped with a connection which has logarithmic poles in Z^i, and we denote by $\Omega^\bullet(\mathcal{I}_{Z^i})$ the corresponding de Rham complex, with logarithmic poles in Z^i. Then

$$\mathbb{R}\Gamma_c(X^i, \Omega^\bullet_{X^i}) \xrightarrow{\sim} \mathbb{R}\Gamma(\bar{X}^i, \Omega^\bullet(\mathcal{I}_{Z^i})).$$

The $\Omega^\bullet(\mathcal{I}_{Z^i})$ form a complex of sheaves on $\bar{X}^\bullet$ which we denote by $\Omega^\bullet(\mathcal{I}_{Z^\bullet})$. We write

$$\mathbb{R}\Gamma_c(X^\bullet, \Omega^\bullet_{X^\bullet}) = T(R\Gamma(\bar{X}^\bullet_{Zar}/\Delta, \Omega^\bullet(\mathcal{I}_{Z^\bullet}))),$$

and denote by $H^i_c(X^\bullet, \Omega^\bullet)$ the cohomology of this complex. We could have also defined the functor $\mathbb{R}\Gamma_c(X^\bullet_{Zar}, \_)$ directly, as the total derived functor of the functor "global sections with compact support." That the two definitions agree follows from the remarks in the previous paragraph. We then have

PROPOSITION (1.10): *In the situations 1,2,3 respectively of (1.5), there exist canonical quasi-isomorphisms*

(1.10.1) $$\mathbb{R}\Gamma(X, \Omega^\bullet_X) \xrightarrow{\sim} \mathbb{R}\Gamma(X^\bullet, \Omega^\bullet_{X^\bullet}),$$

(1.10.2) $$\mathbb{R}\Gamma_Z(X, \Omega^\bullet_X) \xrightarrow{\sim} \mathbb{R}\Gamma_{Z^\bullet}(X^\bullet, \Omega^\bullet_{X^\bullet}),$$

(1.10.3) $$\mathbb{R}\Gamma_c(X, \Omega^\bullet_X) \xrightarrow{\sim} \mathbb{R}\Gamma_c(X^\bullet, \Omega^\bullet_{X^\bullet}).$$

Proof: It is clear from the definitions that we have canonical morphisms of the sort required in the proposition. To see that they are quasi-isomorphisms we can reduce to the case $L = \mathbb{C}$: Our situation is a base change from one defined over a subfield $L' \subset L$, which is finitely generated over $\mathbb{Q}$. Thus, we may replace L by L'. Choosing an embedding $L \hookrightarrow \mathbb{C}$, and base changing by $\otimes_L \mathbb{C}$, we see that we may assume $L = \mathbb{C}$. In this case the result follows from the equality of algebraic and complex analytic de Rham cohomology, and the fact that the latter is isomorphic to Betti cohomology, which satisfies descent for a proper morphisms. More precisely the Betti cohomology analogue of (1.10.1) is [De, 5.3.5], and the analogues of (1.10.2), and (1.10.3) may be deduced from it, as in the proof of (1.6). ∎

(1.11) The results of (1.10) continue to hold, if we no longer assume that X is smooth, and use Hartshorne's definition of de Rham cohomology for a singular scheme [Ha] (see especially pp. 28–29). In this case one defines $\mathbb{R}\Gamma(X, \Omega^\bullet_X)$ as the cohomology of the infinitesimal site of X, and one computes it by choosing local embeddings of X into smooth schemes. Similarly, one defines $\mathbb{R}\Gamma_Z(X, \Omega^\bullet_X)$ and $\mathbb{R}\Gamma_c(X, \Omega^\bullet_X)$ as cohomology of the infinitesimal site with the appropriate support condition.

With these definitions, we get the analogue of (1.10.1), since this theory also commutes with base change [Ha, III 5.2], and can be compared with Betti cohomology [Ha, §4]. The analogues of (1.10.2) and (1.10.3) are then deduced from (1.10.1), as in the proof of (1.6).

(1.12) Suppose we are in Situation 1 of (1.5), and write $X^{\leqslant m} = X^\bullet|_{\Delta[m]}$. We define

$$\mathbb{R}\Gamma(X^{\leqslant m}, \Lambda) = T(R\Gamma(X^{\leqslant m}_{\acute{e}t}/\Delta[m])).$$

There is a natural map

$$\mathbb{R}\Gamma(X^\bullet, \Lambda) \to \mathbb{R}\Gamma(X^{\leqslant m}, \Lambda)$$

which is a quasi-isomorphism in degrees $\leqslant m$. A similar remark holds in the other two situations of (1.5), and also for de Rham cohomology.

§2. The comparison isomorphism for simplicial schemes

(2.1) In this section we work over a complete, discretely valued characteristic 0 field K, with perfect residue field k of characteristic p, and ring of integers $\mathcal{O}_K$. We denote by $K_0 \subset K$ the maximal absolutely unramified subfield of K. We denote by A_{cris}, B_{cris} and B_{st} the usual p-adic period ring of Fontaine [Fon].

We are going to consider pairs $(\mathcal{X}, \mathcal{Z})$ with $\mathcal{X}$ a proper flat $\mathcal{O}_K$-scheme, and $\mathcal{Z} \subset \mathcal{X}$ a Zariski closed subset, which is flat over $\mathcal{O}_K$. We call the pair $(\mathcal{X}, \mathcal{Z})$

semi-stable over $\mathcal{O}_K$ if $\mathcal{X}$ is semi-stable, and $\mathcal{Z} \cup \mathcal{X}_s$ is a normal crossing divisor in $\mathcal{X}$. Here $\mathcal{X}_s$ denotes the special fibre of $\mathcal{X}$.

If $K' \subset K$ is a subfield with K finite over K', and $(\mathcal{X}', \mathcal{Z}')$ is a semi-stable pair over $\mathcal{O}_{K'}$, we consider the $\mathcal{O}_K$-schemes $\mathcal{X} = \mathcal{X}' \otimes_{\mathcal{O}_{K'}} \mathcal{O}_K$ and $\mathcal{Z}$ the Zariski closure of $\mathcal{X}_\eta \times_{\mathcal{X}'} \mathcal{Z}'$ in $\mathcal{X}$, where $\mathcal{X}_\eta$ denotes the generic fibre of $\mathcal{X}$.

We call a pair $(\mathcal{X}, \mathcal{Z})$ **weakly semi-stable** over $\mathcal{O}_K$ if étale locally it comes from a semi-stable pair $(\mathcal{X}', \mathcal{Z}')$ defined over some subfield $K' \subset K$ of finite index, as above. The key point about weakly semi-stable pairs is that they are a rather simple example of logarithmically smooth schemes, with reduced special fibres. In particular they satisfy the hypotheses (LC) of [Fa, p. 28], so we may apply the results of that paper to them (see below).

We write $X = \mathcal{X} \otimes K - \mathcal{Z} \otimes K$, and $X_{\bar{K}} = X \otimes_K \bar{K}$, where $\bar{K}$ denotes the algebraic close of K.

A map between weakly semi-stable pairs $(\mathcal{X}', \mathcal{Z}') \to (\mathcal{X}, \mathcal{Z})$ is a map $f\colon \mathcal{X}' \to \mathcal{X}$ of $\mathcal{O}_K$-schemes such that $f^{-1}(\mathcal{Z}) \subset \mathcal{Z}'$ (on underlying topological spaces). We denote by *WSP* the category of weakly semi-stable pairs.

(2.2) Next we need to introduce crystalline sites. Choose a uniformiser π of $\mathcal{O}_K$, and denote by R the p-adic completion of the divided power envelope of $\mathcal{O}_{K_0}[w]$ with respect to the kernel of the map

$$\mathcal{O}_{K_0}[w] \xrightarrow{w \mapsto \pi} \mathcal{O}_K.$$

(This is denoted by R_V in [Fa], and the variable w is denoted there by t.) R is equipped with a Frobenius induced by the Frobenius on $\mathcal{O}_{K_0}$, and $w \mapsto w^p$.

We regard $\mathcal{X}$ as equipped with its natural log. structure (which is non-trivial along $\mathcal{X}_s \cup \mathcal{Z} \subset \mathcal{X}$), and R as equipped with the log. structure given by $\mathbb{N} \to R$, $1 \mapsto w$. For each positive integer s we write $\mathcal{X}_s = \mathcal{X} \otimes \mathbb{Z}/p^s\mathbb{Z}$, and $R_s = R \otimes \mathbb{Z}/p^s\mathbb{Z}$. In [Fa, p. 51] there is defined the logarithmic crystalline site $(\mathcal{X}_s/R_s)^{\log}_{crys}$. We denote by $(\mathcal{X}/R)^{\log}_{crys}$ the topos obtained by passing to the inverse limit over s. Namely an object in $(\mathcal{X}/R)^{\log}_{crys}$ is a system of sheaves $\{\mathcal{E}_s\}_{s \geq 1}$ with $\mathcal{E}_s$ in $(\mathcal{X}_s/R_s)^{\log}_{crys}$ with $\mathcal{E}_{s+1} \otimes \mathbb{Z}/p^s\mathbb{Z} \xrightarrow{\sim} \mathcal{E}_s$. (We leave it to the reader to define the underlying site of this topos — the logarithmic crystalline site of $\mathcal{X}/R$. In fact a better notation might be $(\hat{\mathcal{X}}/R)^{\log}_{crys}$, where $\hat{\mathcal{X}}$ denotes the completion of $\mathcal{X}$ along its special fibre.)

Write A^0_{cris} for the preimage of $\mathcal{O}_K$ under the usual map $\theta\colon A_{cris} \to \mathcal{O}_{\mathbb{C}_p}$. We may equip A^0_{cris}, and hence A_{cris} and B_{cris}, with an R-algebra structure, as follows. Denote by

$$\underline{\pi} \in \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p$$

an element of the form $(\ldots, \pi^{1/p^2}, \pi^{1/p}, \pi)$. The map $\mathcal{O}_{K_0}[w] \to \mathcal{O}_K$ lifts to a map of $\mathcal{O}_{K_0}$-algebras $\mathcal{O}_{K_0}[w] \to A^0_{cris}$, by sending w to the Teichmüller representative $[\underline{\pi}] \in A^0_{cris}$. As $A^0_{cris} \to \mathcal{O}_K$ sends $[\underline{\pi}]$ to π and has a kernel which is equipped with divided powers, and A^0_{cris} is p-adically complete, we get a map $R \to A^0_{cris}$. This allows us to regard A^0_{cris} with the log. structure given by $\mathbb{N} \to A^0_{cris}$ $1 \mapsto [\underline{\pi}]$ as a formal divided power thickening in the logarithmic crystalline site of $\mathcal{O}_K/R$.

(2.3) We recall the following site defined by Faltings [Fa, p. 27]. Objects are pairs (U, V), where $U \to \mathcal{X}$ is étale, and $V \to U \times_{\mathcal{X}} X_{\bar{K}}$ is finite étale. Maps are pairs of compatible maps $U' \to U$ $V' \to V$, and coverings are pairs of surjective maps. We denote this site by $\mathcal{X}_{Fa}$ (although it depends also on $\mathcal{Z}$). There is morphism of sites $(X_{\bar{K}})_{ét} \to \mathcal{X}_{Fa}$ given by $(U, V) \mapsto V$. Also, we denote by r the morphism of sites $r\colon \mathcal{X}_{Fa} \to \mathcal{X}_{ét}$ given by $U \mapsto (U, U_{\bar{K}})$ for U Zariski open in $\mathcal{X}$.

According to [Fa, p. 50], for every positive integer s, the natural map

$$\mathbb{R}\Gamma(\mathcal{X}_{Fa}, \mathbb{Z}/p^s\mathbb{Z}) \to \mathbb{R}\Gamma(X_{\bar{K},ét}, \mathbb{Z}/p^s\mathbb{Z}) \tag{2.3.1}$$

is a quasi-isomorphism.

We need certain sheaves of rings on $\mathcal{X}_{Fa}$. These are defined in [Fa, pp. 31, 54], but our descriptions differs a little from that of Faltings.

Let

$$\zeta = [\underline{p}] - p \in W := W(\varprojlim_{x \mapsto x^p} \mathcal{O}_{\bar{K}}/p).$$

For positive integers r, s we defined a sheaf "$A_{\rm inf}(\bar{\mathcal{O}})/(\zeta^r, p^s)$" on $\mathcal{X}_{Fa}$ as follows. Let (U, V) be in $\mathcal{X}_{Fa}$, assume that $U = \mathrm{Spec}(H)$ is affine and denote by H' the normalisation of H in V. Consider the site consisting of infinitesimal thickenings $\mathrm{Spec}\, H'/p^sH' \hookrightarrow \mathrm{Spec}\, G$ over $\mathcal{O}_{K_0}/p^s\mathcal{O}_{K_0}$ such that the kernel of $f\colon G \to H'/p^sH'$ satisfies $\ker(f)^r = 0$. Then "$A_{\rm inf}(\bar{\mathcal{O}})/(\zeta^r, p^s)$" is the sheaf associated to the presheaf

$$(U, V) \mapsto H^0_{{\rm inf},r}(H'/p^sH'/\mathcal{O}_{K_0}/p^s\mathcal{O}_{K_0}, \mathcal{O}_{{\rm inf},r})$$

obtained by taking global sections over the site described above. We denote by $A_{\rm inf}(\bar{\mathcal{O}})$ the pro-sheaf $\varprojlim$ "$A_{\rm inf}(\bar{\mathcal{O}})/(\zeta^r, p^s)$", where the inverse limit runs over r and s.

Similarly, we define sheaves "$B^+_{cr}(\bar{\mathcal{O}})/p^s$" and "$B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s$" on $\mathcal{X}_{Fa}$ as follows. "$B^+_{cr}(\bar{\mathcal{O}})/p^s$" is the sheaf associated to the presheaf which attaches to (U, V) the crystalline cohomology of $(H'/p^sH')/(\mathcal{O}_{K_0}/p^s\mathcal{O}_{K_0})$. Let $\bar{\mathcal{O}}$ denote the sheaf attached to the presheaf $(U, V) \mapsto H'$. Then we write "$B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s$" for the subsheaf of $"B^+_{cr}(\bar{\mathcal{O}})/p^s"$ consisting of those sections which map to $H/p^sH \subset H'/p^sH'$ under the natural map $"B^+_{cr}(\bar{\mathcal{O}})/p^s" \to \bar{\mathcal{O}}/p^s\bar{\mathcal{O}}$. A standard argument shows that this map is a surjection. (In the notation of [Fa] this corresponds to the fact that $B^+_{cr}(\bar{R})$ surjects onto $\hat{\bar{R}}$.) In particular, locally on $\mathcal{X}_{Fa}$ $"B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s"$ is a divided power thickening of H/p^sH. We denote by $B^+_{cr}(\bar{\mathcal{O}})$ and $B^{+,0}_{cr}(\bar{\mathcal{O}})$ the pro-sheaves $\varprojlim "B^+_{cr}(\bar{\mathcal{O}})/p^s"$ and $\varprojlim "B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s"$ respectively. We will abuse notation, and drop the " " from the notation above.

These constructions are functorial in the pair $(\mathcal{X}, \mathcal{Z})$. In particular, when $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_K)$, this construction gives the rings A_{cris} and A^0_{cris}, and in general $B^+_{cr}(\bar{\mathcal{O}})$ (resp. $B^{+,0}_{cr}(\bar{\mathcal{O}})$) is a sheaf of A_{cris} (resp. A^0_{cris})-algebras.

(2.4) Let D be a small category, and consider a functor $F\colon D^{\mathrm{op}} \to WSP$. We denote by $(\mathcal{X}^\bullet, \mathcal{Z}^\bullet)$ the image of the functor F, and write $X^\bullet = \mathcal{X}^\bullet \otimes K - \mathcal{Z}^\bullet \otimes K$. (So $X^\bullet$ is the image of a functor from D^{op} to K-schemes.) We may consider various sites on the schemes in $\mathcal{X}^\bullet$ (étale, log. crystalline etc.), and for each such choice we get a topos over D^{op}.

Let $\mathcal{E}$ be a sheaf on $(\mathcal{X}^\bullet/R)^{\log}_{crys}$. Note that for any positive integer s, $\mathcal{E}$ induces a sheaf, again denoted by $\mathcal{E}$, on the log. crystalline topos of the reduction of $(\mathcal{X}^\bullet/R)$ modulo p^s. We attach to $\mathcal{E}$ a sheaf $\mathcal{E}(B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s)$ on $\mathcal{X}^\bullet_{Fa}$, as follows. It is enough to define $\mathcal{E}(B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s)$ on (U,V) such that $B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s$ is a divided power thickening of H/p^sH, since any (U,V) can be covered by neighbourhoods satisfying the above condition. For such (U,V) we define

$$\mathcal{E}(B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s)(U,V) = \mathcal{E}(B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s(U,V)),$$

where $B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s$ is equipped with an R-algebra structure via that on A^0_{cris}.

Evaluation of global sections over each fibre of $\mathcal{X}^\bullet \to D^{\mathrm{op}}$ gives a map

$$\Gamma((\mathcal{X}^\bullet/R)^{\log}_{crys}/D, \mathcal{E}) \to \Gamma(\mathcal{X}^\bullet_{Fa}/D, \mathcal{E}(B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s)). \tag{2.4.1}$$

Hence, we also get a map on total derived functors

$$R\Gamma((\mathcal{X}^\bullet/R)^{\log}_{crys}/D, \mathcal{E}) \to R\Gamma(\mathcal{X}^\bullet_{Fa}/D, \mathcal{E}(B^{+,0}_{cr}(\bar{\mathcal{O}})/p^s)). \tag{2.4.2}$$

In particular, we have the above, when $\mathcal{E} = \mathcal{O}_{(\mathcal{X}^\bullet/R)^{\log}_{crys}}$. Taking the limit over s we may form the composite

$$\begin{aligned} R\Gamma((\mathcal{X}^\bullet/R)^{\log}_{crys}/D, \mathcal{O}_{(\mathcal{X}^\bullet/R)^{\log}_{crys}}) &\to R\Gamma(\mathcal{X}^\bullet_{Fa}/D, B^{+,0}_{cr}(\bar{\mathcal{O}})) \\ &\to R\Gamma(\mathcal{X}^\bullet_{Fa}/D, B^+_{cr}(\bar{\mathcal{O}})). \end{aligned} \tag{2.4.3}$$

(2.5) It follows from the main result of [Fa, §3], that the natural map

$$R\Gamma(\mathcal{X}^\bullet_{Fa}/D, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}}) \to R\Gamma(\mathcal{X}^\bullet_{Fa}/D, A_{\mathrm{inf}}(\bar{\mathcal{O}})) \tag{2.5.1}$$

is an almost quasi-isomorphism, in the sense that the kernels and cokernels of the maps on cohomology are killed by $\mathfrak{n} \subset A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})$, the ideal generated by elements of the form $[\underline{p}^\alpha]$ ($\alpha \in \mathbb{Q}^+$), where $[\underline{p}^\alpha]$ denotes a Teichmüller representative which satisfies $v_p(\theta([\underline{p}^\alpha])) = \alpha$.

Next we have a natural isomorphism (cf. [Fa, p. 54])

$$A_{\mathrm{inf}}(\bar{\mathcal{O}}) \otimes_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})} A_{cris}/p^s \xrightarrow{\sim} B^+_{cr}(\bar{\mathcal{O}})/p^s.$$

Note that, since ζ, p form an $A_{\mathrm{inf}}(\bar{\mathcal{O}})$-regular sequence, and $A_{\mathrm{inf}}(\bar{\mathcal{O}})/(\zeta, p) \xrightarrow{\sim} \bar{\mathcal{O}}/p$ is flat over $\mathcal{O}_{\bar{K}}/p$, we have an isomorphism

$$A_{\mathrm{inf}}(\bar{\mathcal{O}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})} M \xrightarrow{\sim} A_{\mathrm{inf}}(\bar{\mathcal{O}}) \otimes_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})} M$$

for any $A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})$-module M which is killed by some power of the ideal (ζ, p). (I am grateful to the referee for pointing out that; because these rings are not Noetherian, it is not clear whether $A_{\mathrm{inf}}(\bar{\mathcal{O}})$ is actually flat over $A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})$.) In particular, we can apply this to $M = A_{cris}/p^s$.

Thus, applying $\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})} A_{cris}/p^s$ to (2.5.1), we get almost isomorphisms

$$\begin{aligned} R\Gamma(\mathcal{X}^\bullet_{Fa}/D, A_{cris}/p^s) \to R\Gamma(\mathcal{X}^\bullet_{Fa}/D, A_{\mathrm{inf}}(\bar{\mathcal{O}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})} A_{cris}/p^s \\ \xrightarrow{\sim} R\Gamma(\mathcal{X}^\bullet_{Fa}/D, B^+_{cr}(\bar{\mathcal{O}})/p^s). \end{aligned}$$

Here, the last isomorphism may be seen by noting that for each $r \geq ps$ we have

$$\begin{aligned} & R\Gamma(\mathcal{X}^\bullet_{Fa}/D, A_{\mathrm{inf}}(\bar{\mathcal{O}})/(\zeta^r, p^s)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})/(\zeta^r, p^s)} A_{cris}/p^s \\ & \quad \xrightarrow{\sim} R\Gamma(\mathcal{X}^\bullet_{Fa}/D, A_{\mathrm{inf}}(\bar{\mathcal{O}})/(\zeta^r, p^s) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})/(\zeta^r, p^s)} A_{cris}/p^s) \\ & \quad \xrightarrow{\sim} R\Gamma(\mathcal{X}^\bullet_{Fa}/D, A_{\mathrm{inf}}(\bar{\mathcal{O}})/(\zeta^r, p^s) \otimes_{A_{\mathrm{inf}}(\mathcal{O}_{\bar{K}})/(\zeta^r, p^s)} A_{cris}/p^s) \\ & \quad \xrightarrow{\sim} R\Gamma(\mathcal{X}^\bullet_{Fa}/D, B^+_{cr}(\bar{\mathcal{O}})/(\zeta^r, p^s)) \xrightarrow{\sim} R\Gamma(\mathcal{X}^\bullet_{Fa}/D, B^+_{cr}(\bar{\mathcal{O}})/p^s)) \end{aligned}$$

and passing to the inverse limit over r. Finally, passing to the limit over s yields an almost isomorphism

$$R\Gamma(\mathcal{X}^\bullet_{Fa}/D, A_{cris}) \to R\Gamma(\mathcal{X}^\bullet_{Fa}/D, B^+_{cr}(\bar{\mathcal{O}})). \tag{2.5.2}$$

Now assume that $D = \Delta[m]$ for some m. Apply the functor T to (2.4.3) and (2.5.2), and take cohomology. Using the isomorphism (2.3.1) and the almost isomorphism (2.5.2), we get an almost map (i.e., a map in the category obtained by formally inverting almost isomorphisms)

$$(2.5.3)\quad H^i(T(R\Gamma((\mathcal{X}^\bullet/R)^{\log}_{crys}/D,\mathcal{O}_{(\mathcal{X}^\bullet/R)^{\log}_{crys}}))\otimes_R A_{cris}[1/p] \to H^i(T(R\Gamma(\mathcal{X}^\bullet_{\acute{e}t}/D,\mathbb{Z}_p)))\otimes_{\mathbb{Z}_p} A_{cris}[1/p].$$

As in [Fa, pp. 61–62], there is a natural isomorphism

$$(2.5.4)\quad H^i(T(R\Gamma((\mathcal{X}^\bullet_s/\mathcal{O}_{K_0})^{\log}_{crys}/D,\mathcal{O}_{(\mathcal{X}^\bullet_s/\mathcal{O}_{K_0})^{\log}_{crys}}))\otimes_{\mathcal{O}_{K_0}} R \xrightarrow{\sim} H^i(T(R\Gamma((\mathcal{X}^\bullet/R)^{\log}_{crys}/D,\mathcal{O}_{(\mathcal{X}^\bullet/R)^{\log}_{crys}}))),$$

where $\mathcal{X}^\bullet_s = \mathcal{X}^\bullet\otimes_{\mathcal{O}_K} k$. Combining this with (2.5.3), we get an almost map

$$(2.5.5)\quad H^i(T(R\Gamma((\mathcal{X}^\bullet_s/\mathcal{O}_{K_0})^{\log}_{crys}/D,\mathcal{O}_{(\mathcal{X}^\bullet_s/\mathcal{O}_{K_0})^{\log}_{crys}}))\otimes_{K_0} A_{cris}[1/p] \to H^i(T(R\Gamma(X^\bullet_{et}/D,\mathbb{Q}_p)))\otimes_{\mathbb{Q}_p} A_{cris}[1/p].$$

(2.6) The maps in (2.5.3), (2.5.4) and (2.5.5) are compatible with Frobenius actions. Using this one can deduce that the almost map (2.5.5) is actually a map: Note that (2.5.5) is induced by a map from the left hand side of (2.5.5) to the right hand side multiplied by $[\underline{p}^{-\alpha}]$ for every $\alpha > 0$ (here $[\underline{p}^{-\alpha}] = [\underline{p}^{\alpha}]^{-1}$). Thus, it would be enough to show that $\bigcap_{\alpha>0}[\underline{p}^{-\alpha}]A_{cris}[1/p] = A_{cris}[1/p]$. In fact, I do not know if this is true. However, since Frobenius induces an automorphism of the first term in the tensor product on the left hand side of (2.5.5), it is enough to show that $\bigcap_{\alpha>0}[\underline{p}^{-\alpha}]\phi(A_{cris}[1/p]) \subset A_{cris}[1/p]$. Now assume that $\alpha < 1$. Then $a \in A_{cris}[1/p]$ can be written as $a = \sum_{i=0}^{\infty} a_i[\underline{p}]^i/i!$, where $a_i \in W[1/p]$, and we deduce that $[\underline{p}^{-\alpha}]\phi(A_{cris}[1/p]) \subset [\underline{p}^{-\alpha}]W[1/p] + A_{cris}[1/p]$. We are reduced to showing that

$$(2.6.1)\quad \begin{aligned} 0 &= \bigcap_\alpha([\underline{p}^{-\alpha}]W[1/p] + A_{cris}[1/p])/A_{cris}[1/p] \\ &= \bigcap_\alpha[\underline{p}^{-\alpha}]W[1/p]/([\underline{p}^{-\alpha}]W[1/p]\cap A_{cris}[1/p]). \end{aligned}$$

It is easy to check that $\bigcap_\alpha[\underline{p}^{-\alpha}]W = W$. Since $[\underline{p}^{\beta}]$ and p form a regular sequence in W, we have $[\underline{p}^{-\beta}]W \cap W[1/p] = W$ for any $\beta > 0$, so that $[\underline{p}^{-\beta-\alpha}]W \cap [\underline{p}^{-\alpha}]W[1/p] = [\underline{p}^{-\alpha}]W$. It follows that

$$\bigcap_\alpha[\underline{p}^{-\alpha}]W[1/p] = (\bigcap_\alpha[\underline{p}^{-\alpha}]W)[1/p] = W[1/p].$$

It follows that $\bigcap_\alpha[\underline{p}^{-\alpha}]W[1/p]/W[1/p] = 0$, so in order to show (2.6.1) it is enough to show that $[\underline{p}^{-\alpha}]W[1/p] \cap A_{cris}[1/p] = W[1/p]$, or, equivalently,

$W \cap [\underline{p}^\alpha]A_{cris} = [\underline{p}^\alpha]W$. In fact, it is enough to show this for α of the form $\alpha = 1/p^r$ for r a positive integer.

Now the Frobenius induces an automorphism of W. Hence $x \in [\underline{p}^{1/p^r}]W$ if and only if $\phi^r(x) \in [\underline{p}]W$. It follows that it is enough to show that $W \cap [\underline{p}]A_{cris} = [\underline{p}]W$, or equivalently, that the natural map $W/[\underline{p}]W \to A_{cris}/[\underline{p}]A_{cris}$ is an injection. However, this map even has a retraction: Since p and $[\underline{p}]$ form a regular sequence in W, $W/[\underline{p}]W$ is p-torsion free. Moreover, $W/([\underline{p}], p)W \xrightarrow{\sim} \mathcal{O}_{\bar{K}}/p$, hence the kernel of the map $W/[\underline{p}]W \to \mathcal{O}_{\bar{K}}/p$ is equal to (p), and is, in particular, equipped with divided powers. The fibre product $W/[\underline{p}]W \times_{\mathcal{O}_{\bar{K}}/p} \mathcal{O}_{\mathbb{C}_p}$ therefore has the structure of a formal divided power thickening of $\mathcal{O}_{\mathbb{C}_p}$. The required retraction is then induced by the composite

$$A_{cris} \to W/[\underline{p}]W \times_{\mathcal{O}_{\bar{K}}/p} \mathcal{O}_{\mathbb{C}_p} \to W/[\underline{p}]W,$$

where the first map is constructed using the universal property of A_{cris}, and the second is projection onto the first factor.

(2.7) Now [Fa] shows that (2.5.5) tensored by $\otimes_{A_{cris}[1/p]} B_{cris}$ is an isomorphism in the case when D consists of a single object. It follows that our map is also an isomorphism, because the functor T takes acyclic complexes to acyclic complexes.

The first term in the tensor product on the left hand side of (2.5.5) is equipped with a nilpotent monodromy operator N. If we equip the right hand side of (2.5.5) with the natural diagonal $\mathrm{Gal}(\bar{K}/K)$ action, then (2.5.5) becomes Galois equivariant provided we equip the left hand side with the Galois action given by $\exp(\beta(\sigma)N) \otimes \sigma$, where $\sigma \in \mathrm{Gal}(\bar{K}/K)$), and $\beta(\sigma) \in A_{cris}[1/p]$ is defined by $\sigma([\underline{\pi}]) = \beta(\sigma)[\underline{\pi}]$ (see [Fa3, p. 130] for an explanation of this). Finally, if we choose $u \in B_{st}$ such that $\sigma(u) - u = \beta(\sigma)$, and $\phi(u) = pu$, tensor (2.5.5) by $\otimes_{A_{cris}[1/p]} B_{st}$ and compose with the automorphism $\exp(-N \otimes u)$ of the left hand side, then we get an isomorphism

$$(2.7.1)\quad H^i(T(R\Gamma((\mathcal{X}_s^\bullet/\mathcal{O}_{K_0})^{\log}_{crys}/D, \mathcal{O}_{(\mathcal{X}_s^\bullet/\mathcal{O}_{K_0})^{\log}_{crys}}))) \otimes_{K_0} B_{st} \xrightarrow{\sim} H^i(T(R\Gamma(X_{et}^\bullet/D, \mathbb{Q}_p))) \otimes_{\mathbb{Q}_p} B_{st}$$

which is Galois equivariant, when we let $\mathrm{Gal}(\bar{K}/K)$ act on the left hand side via the second factor.

Now consider the case when $D = \Delta_1[m]$. Using essentially the same arguments as above, we obtain the analogue of (2.7.1) with T_1 in place of T. Indeed, the map is constructed in exactly the same way, and the fact that it is an isomorphism follows from the case $D = \Delta[m]$ by "excision."

Finally, consider again the case $D = \Delta[m]$. Using the result of [Fa] on compactly supported cohomology, and the arguments above, we get a Galois equivariant isomorphism

$$(2.7.2)\quad H^i(T(R\Gamma((\mathcal{X}_s^\bullet/\mathcal{O}_{K_0})^{\log}_{crys}/D, \mathcal{J}_{(\mathcal{X}_s^\bullet/\mathcal{O}_{K_0})^{\log}_{crys}}))) \otimes_{K_0} B_{st} \xrightarrow{\sim} H^i(T(R\Gamma_c(X^\bullet_{\acute{e}t}/D, \mathbb{Q}_p))) \otimes_{\mathbb{Q}_p} B_{st}$$

where $\mathcal{J}$ denotes the crystal on $\mathcal{X}_s^\bullet/\mathcal{O}_{K_0}$ which is defined as follows (cf. [Fa pp. 52–53]): it is enough to make a functorial construction of $\mathcal{J}$ on each $\mathcal{X}_s^i$. Locally $(\mathcal{X}_s^i, \mathcal{Z}_s^i)$ lifts to a pair $(\tilde{\mathcal{X}}_s^i, \tilde{\mathcal{Z}}_s^i)$ over $\mathcal{O}_{K_0}$, with $\tilde{\mathcal{X}}_s^i$ log. smooth with trivial log. structure outside $\mathcal{X}_s^i \cup \tilde{\mathcal{Z}}_s^i$, and this lift is unique up to (non-canonical) isomorphism [Ka, 3.14]. The ideal sheaf $\tilde{J}$ of $\tilde{\mathcal{Z}}_s^i \subset \tilde{\mathcal{X}}_s^i$ is equipped with a connection with logarithmic poles along $\tilde{\mathcal{Z}}_s^i$, and this induces the required log. crystal.

(2.8) We want to explain the connection between the crystalline cohomology groups which appeared in (2.5), and de Rham cohomology. Starting with the case $D = \Delta[m]$, and applying $\otimes_R K$ to (2.5.4) (via $R \overset{w\mapsto\pi}{\longrightarrow} K$), one sees that

$$(2.8.1)\quad H^i(T(R\Gamma((\mathcal{X}_s^\bullet/\mathcal{O}_{K_0})^{\log}_{crys}/D, \mathcal{O}_{(\mathcal{X}_s^\bullet/\mathcal{O}_{K_0})^{\log}_{crys}}))) \otimes_{\mathcal{O}_{K_0}} K \xrightarrow{\sim} H^i(T(R\Gamma((\mathcal{X}^\bullet/R)^{\log}_{crys}/D, \mathcal{O}_{(\mathcal{X}^\bullet/R)^{\log}_{crys}}))) \otimes_R K.$$

However, a direct calculation with de Rham complexes shows that the right hand side is computed by taking the de Rham complex on $(\mathcal{X}^i \otimes K)^{an}$ with logarithmic poles along $(\mathcal{Z}^i \otimes K)^{an}$, forming a double complex, and taking the cohomology of the associated single complex. Here, $(\mathcal{X}^i \otimes K)^{an}$ and $(\mathcal{Z}^i \otimes K)^{an}$ denote the associated analytic spaces. By rigid analytic GAGA, the complex we have obtained is quasi-isomorphic to the analogous complex built from logarithmic de Rham complexes on the algebraic schemes $\mathcal{X}^i \otimes K$, and the cohomology of these computes the de Rham cohomology of $X^i = \mathcal{X}^i \otimes K - \mathcal{Z}^i \otimes K$ by [De2, 3.13]. Thus the right hand side of (2.8.1) is quasi-isomorphic to $H^i(X^\bullet, \Omega^\bullet_{X^\bullet})$. Under these isomorphisms (2.8.1) becomes the simplicial version of the Hyodo–Kato isomorphism [HK, 5.1]. Using (2.7.1), we obtain a Galois equivariant isomorphism

$$(2.8.2)\quad H^i(X^\bullet, \Omega^\bullet_{X^\bullet}) \otimes_{K_0} B_{st} \xrightarrow{\sim} H^i_{\acute{e}t}(X^\bullet_{\acute{e}t}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} \otimes_{K_0} K.$$

Using the variant for $D = \Delta_1[m]$, we get in a similar way

$$(2.8.3)\quad H^i_{Z^\bullet}(X^\bullet, \Omega^\bullet_{X^\bullet}) \otimes_{K_0} B_{st} \xrightarrow{\sim} H^i_{\acute{e}t,Z}(X^\bullet_{\acute{e}t}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} \otimes_{K_0} K.$$

Finally, using (2.7.2) and the characterisation of de Rham cohomology with compact support given in (1.9) we get

$$(2.8.4)\quad H^i_c(X^\bullet, \Omega^\bullet_{X^\bullet}) \otimes_{K_0} B_{st} \xrightarrow{\sim} H^i_c(X^\bullet_{\acute{e}t}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{st} \otimes_{K_0} K.$$

§3. Potential semi-stability of étale cohomology

Before proving results on étale cohomology we need the following

PROPOSITION (3.1): *Let X be a proper K-scheme, and $Z \subset X$ a closed subset. Then there exists a proper flat $\mathcal{O}_K$-scheme $\mathcal{X}$ and a dense, open immersion $X \hookrightarrow \mathcal{X}$. Let $\mathcal{Z}$ denote the closure of Z in $\mathcal{X}$.*

Let m be a positive integer. For any $\mathcal{X}$ as above, there exists a finite extension K' of K such that, after replacing X, Z and $\mathcal{X}$ by $X \otimes_K K', Z \otimes_K K'$, and $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$, and $\mathcal{Z}$ by the closure of $Z \otimes_K K'$ in $\mathcal{X}$, we have the following situation:

There exists a proper hypercovering $\mathcal{X}^\bullet \to \mathcal{X}$ such that, if p_i: $\mathcal{X}^i \to \mathcal{X}$ denotes a piece of the hypercovering with $i \leqslant m$, then $(\mathcal{X}^i, p_i^{-1}(\mathcal{Z})^{\mathrm{red}})$ is weakly semi-stable.

Proof: The existence of $\mathcal{X}$ is Nagata's theorem. To construct the required hypercovering we use the result of de Jong [deJ, 6.5]. This says that for any $(\mathcal{X}, \mathcal{Z})$ as above, after a finite extension of scalars K'/K (replacing $\mathcal{X}$ by $\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$, as above), we can find a proper flat $\mathcal{O}_K$-scheme, and a proper surjective, generically finite map $p : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ has semi-stable reduction, and $p^{-1}(\mathcal{Z})^{\mathrm{red}} \cup \mathcal{X}'_s$ is a normal crossings divisor.

The proposition now follows by repeatedly applying de Jong's result, first to $(\mathcal{X}, \mathcal{Z})$, as above, then to suitable fibre powers of pieces of the hypercovering already constructed (after omitting components supported in the special fibre). We refer to [SD, 5.1.3] for details. If we only require that $(\mathcal{X}^i, p_i^{-1}(\mathcal{Z})_{\mathrm{red}})$ is weakly semi-stable for $i \leqslant m$ for some positive integer m, then we can assume that the hypercovering $\mathcal{X}^\bullet$ is defined over $\mathcal{O}_{K'}$ with K' a finite extension of K. ∎

THEOREM (3.2): *Let X be any finite type K-scheme, and $Y \subset X$ a closed subset. Then for $i \in \mathbb{N}$, the representations of $\mathrm{Gal}(\bar{K}/K)$ on the étale cohomology groups $H^i(X_{\bar{K}}, \mathbb{Q}_p), H^i_c(X_{\bar{K}}, \mathbb{Q}_p), H^i_Y(X_{\bar{K}}, \mathbb{Q}_p)$ are all potentially semi-stable.*

Proof: By Nagata's theorem there is a proper K-scheme $\bar{X}$, which admits X as a dense, open subscheme. To see the claim for $H^i(X_{\bar{K}}, \mathbb{Q}_p)$ (resp. $H^i_c(X_{\bar{K}}, \mathbb{Q}_p)$) we apply (3.1) to $\bar{X}$ and $Z = \bar{X} - X$. Choose $m \geq 2d + 1$, where d is the maximum of the dimensions of the irreducible components of $\mathcal{X}$. Then the result follows from (2.7.1) and (1.6.1) (resp. (2.7.2) and (1.6.3)) keeping in mind the remark of (1.12).

Finally, for $H^i_Y(X_{\bar{K}}, \mathbb{Q}_p)$ choose $\bar{X}$ as before, and set $Z = \bar{X} - X$, and $\bar{Y}$ equal to the closure of Y in $\bar{X}$. We apply (3.1) to $Z \cup \bar{Y} \subset \bar{X}$. Now the result follows from (1.6.2), and the remark following (2.7.1) ∎

THEOREM (3.3): *Under the assumptions of (3.2), write $B = B_{st} \otimes_{K_0} \bar{K}$. There are canonical isomorphisms*

(3.4.1) $$H^i(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B \xrightarrow{\sim} H^i(X, \Omega^\bullet_X) \otimes_K B,$$

(3.4.1) $$H^i_c(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B \xrightarrow{\sim} H^i_c(X, \Omega^\bullet_X) \otimes_K B,$$

(3.4.1) $$H^i_{Y_{\bar{K}}}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B \xrightarrow{\sim} H^i_Y(X, \Omega^\bullet_X) \otimes_K B.$$

In particular, the same is true if we instead take $B = B_{DR}$.

Proof: This follows from (1.10), (2.8), and the same arguments as in the proof of (3.2). ∎

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